Tutorial 3

Q.1

Evaluate the following integrals.

(a) $\int_0^3 \int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\, dy\, dx$

(b) $\int_0^1 \frac{x^{-a}-x^{-b}}{\log x} dx := \lim_{t\to 0} \int_t^{1-t} \frac{x^{-a}-x^{-b}}{\log x} dx$, where $1<a<b$.

Solution

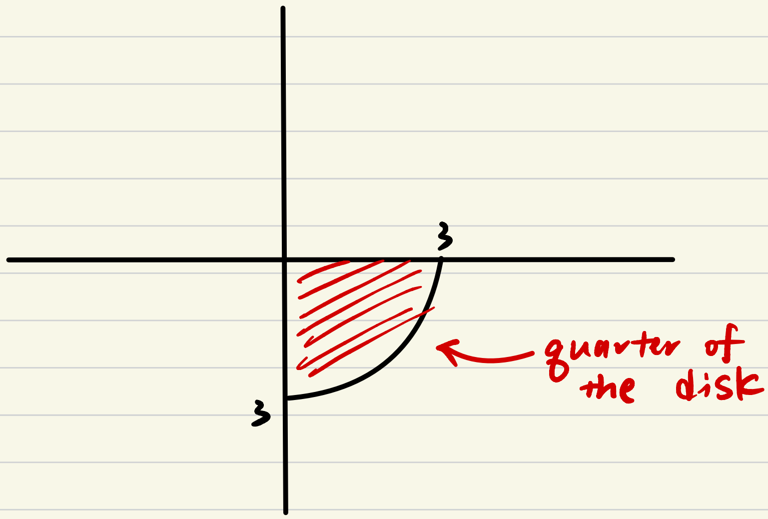
(a) The region of integration:

$$0 \le x \le 3$$

$$-\sqrt{9-x^2} \le y \le 0$$

$$-\sqrt{9-x^2} = y$$

$$9 = x^2+y^2$$



It is easier to work with polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\,dy\,dx$$

$$= \int_{-\frac{\pi}{2}}^{0} \int_0^3 e^{r^2}\, r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} \frac{1}{2}(e^9-1)\,d\theta$$

$$= \frac{\pi}{4}(e^9-1)$$

(b) Fix $t \in (0,1)$.

By the fundamental thm of calculus,

$$x^{-a} - x^{-b} = \int_b^a \frac{d}{dy}(x^{-y})\,dy = \int_a^b (\log x)\, x^{-y}\,dy$$

$$\therefore \int_t^{1-t} \frac{x^{-a}-x^{-b}}{\log x}\,dx$$

$$= \int_t^{1-t} \frac{1}{\log x}\left(\int_a^b (\log x)\, x^{-y}\,dy\right) dx$$

$$= \int_t^{1-t} \int_a^b x^{-y}\,dy\,dx$$

$$= \int_a^b \int_t^{1-t} x^{-y}\,dx\,dy \quad \text{(Fubini)}$$

$$= \int_a^b \left(\frac{(1-t)^{-y+1}}{-y+1} - \frac{t^{-y+1}}{-y+1}\right) dy \quad (y \geq a > 1 \text{, so won't see } x^{-1})$$

This is still not computable, but if we can do the following, then we are done.

$$\lim_{t\to 0}\int_a^b \frac{(1-t)^{-y+1}}{-y+1} - \frac{t^{-y+1}}{-y+1}\,dy$$

$$= \int_a^b \lim_{t\to 0}\left(\frac{(1-t)^{-y+1}}{-y+1} - \frac{t^{-y+1}}{-y+1}\right)dy \quad (?)$$

$$= \int_a^b \frac{1}{-y+1}\,dy$$

$$= \left[-\log(-y+1)\right]_a^b$$

$$= \log\left(\frac{-a+1}{-b+1}\right)$$

We need to justify the equality in the line (?).

More generally, we have the following thm.

<u>Thm 1</u>: Let $f:[a,b]\times[c,d]\longrightarrow\mathbb{R}$ be a cont. func., where $|a|,|b|<\infty$.

Then $\forall t_0\in[c,d]$, we have

$$\lim_{t\to t_0}\int_a^b f(y,t)\,dy = \int_a^b \lim_{t\to t_0} f(y,t)\,dy$$

$$= \int_a^b f(y,t_0)\,dy$$

Proof:

$$\left| \int_a^b f(y,t)\,dy - \int_a^b f(y,t_0)\,dy \right|$$

$$\leq \int_a^b |f(y,t) - f(y,t_0)|\,dy$$

uniform because we don't want δ to depend on y

By the uniform cont. of f, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $|t-t_0| < \delta$, then $|f(y,t) - f(y,t_0)| < \frac{\varepsilon}{2(b-a)}$ $\forall y \in [a,b]$

$\Rightarrow$ If $|t - t_0| < \delta$, then

$$\int_a^b |f(y,t) - f(y,t_0)|\,dy \leq \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2} < \varepsilon.$$

$\square$

Remark: The proof above only works for $|a|, |b| < \infty$.

Q.2

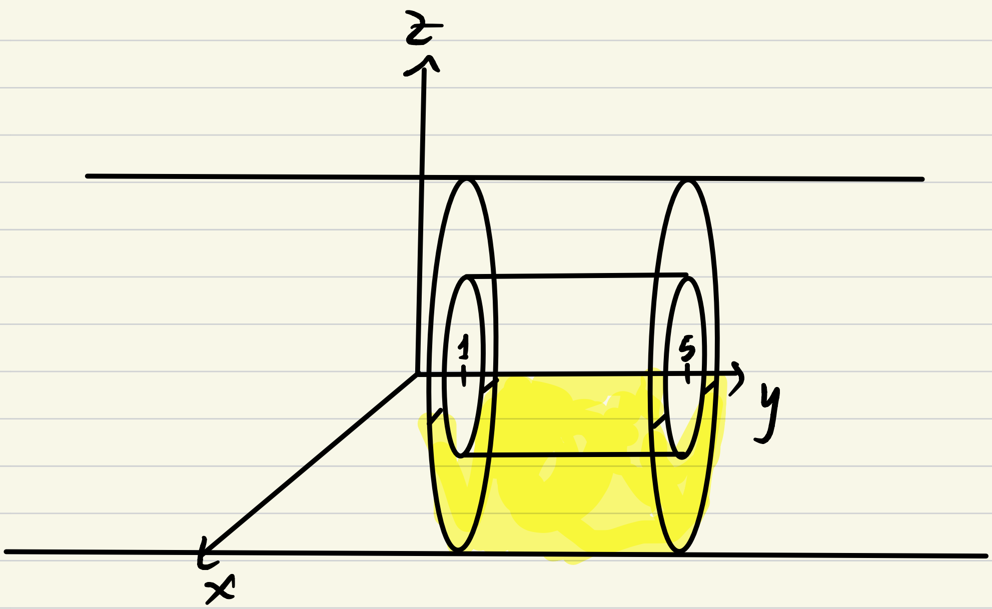
Evaluate the following integrals.

(a) $\int_E e^{-x^2-z^2}\,dV$, where E is the region between two cylinders $x^2+z^2=4$ & $x^2+z^2=9$ with $1\le y\le 5$, $z\le 0$.

(b) $\int_F 3z\,dV$, where F is the region inside both $x^2+y^2+z^2=1$ & $z=\sqrt{x^2+y^2}$.

Solution:

(a) The region of integration:



We should use cylindrical coordinates:

$$x=r\cos\theta,\quad y=y,\quad z=r\sin\theta.$$

$$dV=r\,dr\,d\theta\,dy$$

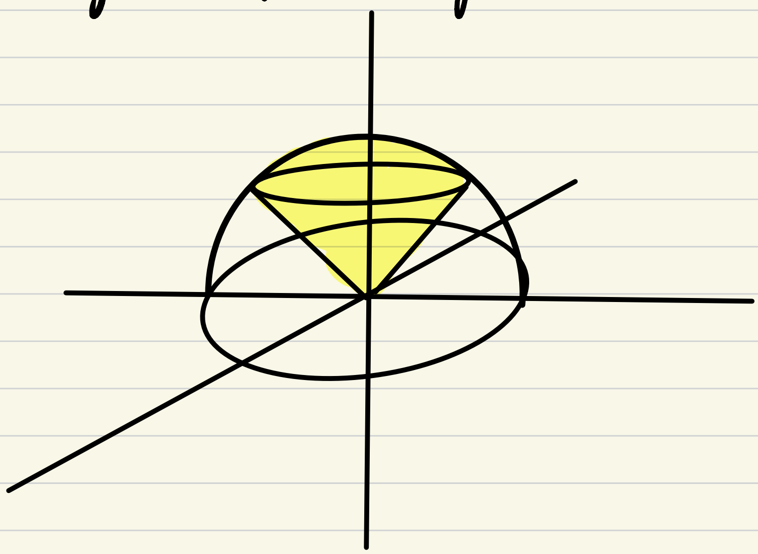
$$\int_E e^{-r^2-z^2}\,dV$$

$$= \int_1^5 \int_{-\pi}^0 \int_2^3 e^{-r^2}\, r\,dr\,d\theta\,dy$$

$$= 4 \cdot \pi \cdot \left[-\frac{1}{2} e^{-r^2}\right]_2^3$$

$$= 2\pi\left(e^{-4} - e^{-9}\right)$$

(b) The region of integration:



We should use spherical coordinates.

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$

$0 \le \rho \le 1$, $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$.

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\int_E 3z\,dV$$

$$= \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} 3\rho\cos\phi\,(\rho^2\sin\phi)\,d\theta\,d\phi\,d\rho$$

$$= 3\pi \int_0^1 \int_0^{\frac{\pi}{4}} \rho^3 \sin 2\phi\,d\phi\,d\rho$$

$$= 3\pi \left[-\frac{\cos 2\phi}{2}\right]_0^{\frac{\pi}{4}} \left[\frac{\rho^4}{4}\right]_0^1$$

$$= 3\pi\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{3\pi}{8}$$